MATH 4010 | Tutorial 9 | 19/03/2015

Eg 1. Let M be a subset of a Hilbert space X. What is $(M^\perp)^\perp$?

Solu: $(M^\perp)^\perp = \overline{\text{span} M}$. Indeed, by the definition of orthogonal complement

$x \in M^\perp \Leftrightarrow x \perp M \Leftrightarrow x \perp \text{span} M \Leftrightarrow x \perp \overline{\text{span} M} \Leftrightarrow x \in \overline{\text{span} M}^\perp$

So, $M^\perp = \overline{\text{span} M}^\perp$.

To prove $(M^\perp)^\perp = \overline{\text{span} M}$, it sufficies to show that ~~$(M^\perp)^\perp$~~

$(\overline{\text{span} M}^\perp)^\perp = \overline{\text{span} M}$.

This follows from the fact that $(N^\perp)^\perp = N$, if N is closed.

Pf of the fact:

$\forall x \in N \Rightarrow x \perp N^\perp \Rightarrow x \in (N^\perp)^\perp \Rightarrow N \subset (N^\perp)^\perp$

Suppose $N \subsetneqq (N^\perp)^\perp$. Then N is a proper closed subspace of $(N^\perp)^\perp$.

By the orthogonal decomposition Thm, $\forall x \in (N^\perp)^\perp \setminus N$

$x = y + z$ with $y \in N$, $z \in N^\perp$ and $z \neq 0$.

Since $y \in N \subset (N^\perp)^\perp$, then $z \doteq x - y \in (N^\perp)^\perp$

Thus $z \in (N^\perp)^\perp \cap N^\perp = \{0\}$, A contradiction, so $(N^\perp)^\perp = N$.

Eg 2. Let X be a Hilbert space, $M \subset X$, $x \in X$.

If M is a closed convex subset, then $\exists!\, y \in M$ s.t.

$$\inf_{\tilde{y} \in M} \|x - \tilde{y}\| = \|x - y\|$$

Pf: Set $\delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\|$. Then by the definition of infimum, there exist

$\delta_n \to \delta$ as $n \to +\infty$ where $\delta_n = \|x - y_n\|$, $y_n \in M$.

Now, we claim that $\{y_n\}$ is a Cauchy sequence.

Set $z_n = x - y_n$, then $\|z_n\| = \delta_n$ and

$\|z_n + z_m\| = \|x - y_n + x - y_m\| = \|2x - (y_n + y_m)\| = 2\|x - \frac{1}{2}(y_n + y_m)\| \geq 2\delta$,

Since M is convex, $\frac{1}{2}(y_n + y_m) \in M$. ~~so $\|z_n + z_m\| \geq 2\delta$~~

Note that $z_n - z_m = y_m - y_n$

$\|y_n - y_m\|^2 = \|z_n - z_m\|^2 = -\|z_n + z_m\|^2 + 2(\|z_n\|^2 + \|z_m\|^2)$ by parallelogram equality

$\leq -(2\delta)^2 + 2(\delta_n^2 + \delta_m^2) \to 0$ as $n, m \to +\infty$

Therefore $\{y_n\}$ is a Cauchy sequence in Hilbert space X ~~s.t.~~

which implies $\exists\, y \in X$ s.t. $y_n \to y$ in X. $y \in M$, since M is closed.

Furthermore, $\|x-y\| \leq \|x-y_n\| + \|y_n - y\| = \delta_n + \|y_n - y\| \to \delta$ as $n \to \infty$

Hence, $\|x-y\| = \delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\|$.

Eg 3. (Characterization of minimizing vector in Hilbert space)

Let M be a closed convex subset of Hilbert space X, $x \in X$.

Then y is the minimizing vector of M if and only if

$$\mathrm{Re}(x-y, y-z) \geq 0, \quad \forall z \in M$$

Pf: $\forall z \in M$, define $\varphi_z(t) = \|x - tz - (1-t)y\|^2$, $t \in [0,1]$.

Then, y is the minimizing vector iff $\varphi_z(t) \geq \varphi_z(0)$, $\forall z \in M$, $\forall t \in [0,1]$ $(*)$

Note that

$$\varphi_z(t) = \|(x-y) + t(y-z)\|^2 = \|x-y\|^2 + 2t\,\mathrm{Re}\langle x-y, y-z\rangle + t^2\|y-z\|^2$$

Then, $\varphi_z'(0) = 2\mathrm{Re}(x-y, y-z)$

and $\varphi_z(t) - \varphi_x(0) = \varphi_z'(0)t + \|y-z\|^2 t^2 \geq 0$, $\forall t \in [0,1]$

iff $\varphi_z'(0) \geq 0$, i.e. $2\mathrm{Re}\langle x-y, y-z\rangle \geq 0$.

Remark: If M is a closed subspace of Hilbert space,

then y is the minimizing vector of M iff $x - y \perp M$.

In fact, if M is a closed subspace, by eg 3. $y - z =: w \in M$

$\mathrm{Re}(x-y, w) \geq 0 \quad \forall w \in M$

$\Rightarrow \mathrm{Re}(x-y, -w) \geq 0 \Rightarrow \mathrm{Re}(x-y, w) \leq 0$ $\Big\} \Rightarrow \mathrm{Re}(x-y, w) = 0$

$\Rightarrow \mathrm{Re}(x-y, iw) = 0$

$\Downarrow$

$\mathrm{Im}(x-y; w) = 0$

$\Downarrow$

$\langle x-y, w\rangle = 0$.